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1984 J. Phys. A: Math. Gen. 17 3477

(http://iopscience.iop.org/0305-4470/17/18/017)

J. Phys. A: Math. Gen. 17 (1984) 3477–3485. Printed in Great Britain

$\hbar$-smooth quantum solutions: A semiclassical method

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Received 17 April 1984

Abstract. The $\hbar \to 0$ semiclassical expansion of Wigner and Kirkwood is obtained for quantum systems of finitely many particles. In a d-dimensional Euclidean space $\mathbb{R}^d$ without boundaries quantum systems defined by a Hamiltonian, H, given as the sum of the negative Laplacian perturbed by a potential $v(x)$ are considered. The semiclassical behaviour of the kernels of the semigroup family of operators $\{e^{-zH}: \text{Re } z \geq 0\}$ is determined in terms of an asymptotic expansion in the variable $\hbar$. The order of the expansion is proportional to the number of bounded derivatives $v(x)$ supports. The expansion is uniform in $\mathbb{R}^d \times \mathbb{R}^d$ and accompanied by explicit bounds for the error term. The results are obtained for potentials $v(x)$ that are Fourier images of complex bounded measures.

1. Introduction

In this paper explicit asymptotic expansions with remainder term bounds are constructed for the type of semiclassical approximation originating in the work of Wigner (1932) and Kirkwood (1933). In spite of the fact that the semiclassical expansion of Wigner and Kirkwood has been in active use for more than 50 years, a rigorous derivation of the asymptotic nature of this approximation (together with an error estimate) has been lacking.

Suppose H is the generator of time evolution for the N-body problem in non-relativistic quantum mechanics in a d-dimensional Euclidean space $\mathbb{R}^d$. Let x be the generic point in $\mathbb{R}^d$ that determines the position of all N particles. Take $v(x)$ to be the real-valued local potential of the system, then H is the self-adjoint extension in $L^2(\mathbb{R}^d)$ of the quadratic elliptic differential operator

$$H_{(x)} = -q\Delta_x + v(x). \tag{1.1}$$

Here Δ_x is the Laplacian in $\mathbb{R}^d$. If each particle moves in three dimensions then $d = 3N$. In terms of the rationalised value of Planck's constant $\hbar$ and the mass m of each particle the variable q denotes the quantum scale factor

$$q = \hbar^2/2m. \tag{1.2}$$

Throughout this paper it is assumed that $v(x)$ is uniformly bounded in $\mathbb{R}^d$. This suffices to ensure that H is bounded from below. Consider the analytic semigroup defined by H. Let z take values in the open right-half plane $D \subset \mathbb{C}$. The family of bounded operators on $L^2(\mathbb{R}^d)$ given by

$$\{e^{-zH}: z \in D\} \tag{1.3}$$

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0305-4470/84/183487+05\$02.25

defines the analytic semigroup induced by H. The operators e^{-zH} are integral operators with associated Carleman kernels $U(x, y; z, q)$ (Simon 1982, Osborn and Fujiwara 1983). If z is restricted to the positive real axis ($z = \beta > 0$) and if $v(x)$ is appropriately smooth then the kernels $U(x, y; \beta, q)$ are the fundamental solutions to the heat transport equation—or, as it is often referred to in statistical mechanics, the Bloch equation. The value of β is proportional to the inverse temperature of the system. Whereas if z belongs to the boundary of D, the imaginary z axis, then one obtains the unitary time evolution operators. Let $t \in \mathbb{R}$ represent time displacement; then the kernels $U(x, y; \mathrm{i}t/\hbar, q)$ are the fundamental solutions of the time-dependent Schrödinger equation (Osborn *et al* 1984).

The objective of this study is to obtain the $q \to 0$ behaviour of the analytic semigroup kernels $U(x, y; z, q)$ realised in terms of an appropriate asymptotic expansion. The basic idea is that $U(x, y; z, q)$ admits a factorisation into a term with a rapid oscillation as $q \to 0$, multiplied by a function that is slowly varying for $q \simeq 0$. Specifically,

$$U(x, y; z, q) = U_0(x, y; z, q) F(x, y; z, q), \tag{1.4}$$

where

$$U_0(x, y; z, q) = \exp(-|x-y|^2/4qz)/(4\pi qz)^{d/2}. \tag{1.5}$$

If H_0 is the self-adjoint extension of $-q\Delta_x$ then $U_0(x, y; z, q)$ is the standard diffusion kernel associated with e^{-zH_0}. Clearly $U_0(x, y; z, q)$ has an essential singularity at $q = 0$. Nevertheless it will be proved that the function $F(x, y; z, q)$ admits an M-term asymptotic expansion in powers of q.

2. Fourier image potentials

We introduce a class of potentials suitable for describing the N-body problem. Let $*$ denote complex conjugation. A complex bounded measure μ defined on the Borel field $\mathscr{B}$ on $\mathbb{R}^d$ is said to satisfy the reflection property if for all measurable sets $e \in \mathscr{B}$,

$$\mu(-e) = \mu(e)^*. \tag{2.1}$$

Let $\mathscr{M}^*(\mathbb{R}^d)$ be all complex bounded measures on $\mathscr{B}$ satisfying the reflection property. For each $\mu \in \mathscr{M}^*(\mathbb{R}^d)$ define the potential v: $\mathbb{R}^d \to \mathbb{R}$ by

$$v(x) = \int_{\mathbb{R}^d} \exp(\mathrm{i}k \cdot x)\, \mathrm{d}\mu(k). \tag{2.2}$$

Here $k \cdot x$ denotes the scalar product in $\mathbb{R}^d$. The reflection property ensures that $v(x)$ is real valued. Let the set $\mathscr{F}^*$ be the Fourier image of $\mathscr{M}^*(\mathbb{R}^d)$,

$$\mathscr{F}^* = \left\{ v(x) = \int_{\mathbb{R}^d} \exp(\mathrm{i}k \cdot x)\, \mathrm{d}\mu(k) : \mu \in \mathscr{M}^*(\mathbb{R}^d) \right\}. \tag{2.3}$$

The elements of the spaces $\mathscr{F}^*$ and $\mathscr{M}^*(\mathbb{R}^d)$ are in a one-to-one correspondence. This is a consequence of the uniqueness of the transform (2.2) that asserts $v = 0$ if and only if $\mu = 0$ (Rudin 1961). A convenient norm for this pair of spaces is provided by the total variation $|\mu|$ of μ. By setting

$$\|\mu\| = |\mu|(\mathbb{R}^d), \tag{2.4}$$

both $\mathscr{F}^*$ and $\mathscr{M}^*(\mathbb{R}^d)$ become Banach spaces if norm (2.4) is attached.

A modification of $\mathscr{F}^*$ leads to useful estimates of the partial derivatives of $v(x)$. Let M be a positive integer. Define a subset of $\mathscr{F}^*$ by

$$\mathscr{F}^*_M = \left\{ v \in \mathscr{F}^* : \int_{\mathbb{R}^d} |k|^n \, \mathrm{d}|\mu|(k) < \infty \text{ for } n = 0, 1, \ldots, M \right\}. \tag{2.5}$$

If $v \in \mathscr{F}^*_M$ let the number $K = K(v, M)$ be the smallest positive constant such that

$$\int_{\mathbb{R}^d} |k|^n \, \mathrm{d}|\mu|(k) \leq K^n \|\mu\|, \qquad n = 0, 1, \ldots, M. \tag{2.6}$$

The number K is called *the bound constant of the potential v in the space $\mathscr{F}^*_M$*. The utility of these definitions is seen in the following bound. If D^α_x is a partial derivative in $\mathbb{R}^d$ with a multi-index α of length $|\alpha|$, then for all x

$$|(D^\alpha_x v)(x)| \leq K^{|\alpha|} \|\mu\|, \qquad |\alpha| \leq M. \tag{2.7}$$

The class of potentials $\mathscr{F}^*$ is appropriate for treating the N-body problem since there is no assumption of decay as $|x| \to \infty$. Other problems of physical interest lie in this class. For example, scattering from a localised impurity in a crystal lattice and the behaviour of a particle in a continuous random potential. This class of potentials was introduced by Ito (1961) to study the Feynman path integral representations of $\mathrm{e}^{-\mathrm{i}tH}$. Later Albeverio and Høegh-Krohn (1976) have used $\mathscr{F}^*$ for the same purpose.

3. Constructive representation of $U(x, y; z, q)$

This section states the form of the constructive series representation of the semigroup kernels found recently by Osborn and Fujiwara (1983), hereafter OF. First we introduce some of the notation and the simple functions that appear ubiquitously in this representation. For $i = 1, \ldots, n$ let $\xi_i \in [0, 1]$ and $k_i \in \mathbb{R}^d$, set

$$\theta(\xi_i, \xi_j) = \xi_<(1 - \xi_>) \tag{3.1}$$

where $\xi_<$ is the minimum of ξ_i and ξ_j, and $\xi_>$ is the maximum. Define polynomials in k_i by

$$a_n(\xi_1, \ldots, \xi_n; k_1, \ldots, k_n) = \sum_{l,m=1}^{n} \theta(\xi_l, \xi_m) k_l \cdot k_m, \tag{3.2}$$

$$b_n(\xi_1, \ldots, \xi_n; k_1, \ldots, k_n) = \sum_{l=1}^{n} [(1 - \xi_l)x + \xi_l y] \cdot k_l. \tag{3.3}$$

The following n-fold multiple integrals we abbreviate by

$$\int_>^1 \mathrm{d}^n\xi \equiv \int_{1 \geq \xi_1 \geq \ldots \geq \xi_n \geq 0} \cdots \int \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_n, \tag{3.4}$$

$$\int \mathrm{d}^n\mu \equiv \int \ldots \int \mathrm{d}\mu(k_1) \ldots \mathrm{d}\mu(k_n). \tag{3.5}$$

The subsequent definition, lemma and proposition state the form of the constructive representation for $U(x, y; z, q)$.

Definition. Let $v \in \mathscr{F}^*$. For each $(z, q) \in \bar{D} \times \mathbb{R}^+$, let $F(\cdot, \cdot; z, q)$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be the function given by the series

$$F(x, y; z, q) = 1 + \sum_{n=1}^{\infty} B_n(x, y; z, q) \tag{3.6}$$

where

$$B_n(x, y; z, q) = (-z)^n \int_{>}^{1} \mathrm{d}^n\xi \int \mathrm{d}^n\mu \exp(-zqa_n + \mathrm{i}b_n). \tag{3.7}$$

Lemma 1. Let $v \in \mathscr{F}^*$.

(i) Take Σ to be an arbitrary compact subset of $\bar{D}$. The series (3.6) is absolutely and uniformly convergent in $\mathbb{R}^d \times \mathbb{R}^d \times \Sigma \times \mathbb{R}^+$. The sum of (3.6), $F(x, y; z, q)$, has the bound

$$|F(x, y; z, q)| \leqslant \exp(|z| \, \|\mu\|). \tag{3.8}$$

(ii) For each $(x, y, q) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$, $F(x, y; z, q)$ is holomorphic in D and continuous in $\bar{D}$. For each $z \in \bar{D}$, $F(x, y; z, q)$ is continuous in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$.

Proposition. Let $v \in \mathscr{F}^*$. Define $U(x, y; z, q)$ by equation (1.4).

(i) If $(z, q) \in D \times \mathbb{R}^+$ and $f \in L^2(\mathbb{R}^d)$, then for a.a. $x \in \mathbb{R}^d$

$$(\mathrm{e}^{-zH}f)(x) = \int_{\mathbb{R}^d} \mathrm{d}y \, U(x, y; z, q) f(y). \tag{3.9}$$

(ii) Suppose $t \neq 0$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then for a.a. $x \in \mathbb{R}^d$

$$(\mathrm{e}^{-\mathrm{i}tH}f)(x) = \int_{\mathbb{R}^d} \mathrm{d}y \, U(x, y; \mathrm{i}t, q) f(y). \tag{3.10}$$

Lemma 1 and the proposition are proved in OF. Related existence results for time-evolution kernels have been obtained by Fujiwara (1979, 1980), Kitada (1980, 1982), and Zelditch (1983). The constructive representation for the kernels of e^{-zH}, given in lemma 1 and the proposition, have been recently generalised to include the spin-dependent N-body problem, Osborn *et al* (1984).

4. The semiclassical expansion

Let the linear path in $\mathbb{R}^d$ between x and y parametrised by $\xi \in [0, 1]$ be represented by

$$\hat{\xi} = (1 - \xi)x + \xi y. \tag{4.1}$$

M will always denote a positive integer and be equal to the number of terms appearing in the semiclassical expansion. In the following we break up the results about the asymptotic expansion of $F(x, y; z, q)$ into two parts. The lemma concerns the definition and behaviour of the coefficient functions in the expansion. The theorem summarises the structure of the expansion and states the remainder term bounds.

Lemma 2. Suppose $v \in \mathscr{F}^*_{2(M+1)}$ and let K be the associated bound constant of v in the space $\mathscr{F}^*_{2(M+1)}$. Define for $m = 1, \ldots, M$ the coefficient functions $T_m(x, y; z)$:

$\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$ by the convergent series

$$T_m(x, y; z) = \exp\left(z \int_0^1 d\xi\, v(\hat{\xi})\right) \sum_{n=1}^{\infty} \frac{(-z)^{n+m}}{m!} \int_>^1 d^n\xi \int d^n\mu (a_n)^m\, e^{ib_n}. \tag{4.2}$$

If Σ is an arbitrary compact subset of $\mathbb{C}$, then the series (4.2) is uniformly and absolutely convergent in $\mathbb{R}^d \times \mathbb{R}^d \times \Sigma$ for $m = 1, \ldots, M$. For each $m \leq M$ and each fixed coordinate pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, $T_m(x, y; z)$ is an entire function of z. For each fixed $z \in \mathbb{C}$, $T_m(x, y; z)$ is jointly continuous in x, y and has the x, y uniform bound

$$|T_m(x, y; z)| \leq \frac{|z|^{m+1}}{m!} \left(\frac{K^2}{4}\right)^m \exp[|z|\,\|\mu\|(1 + \exp 2(m+1)], \tag{4.3}$$

for $m = 1, \ldots, M$.

Theorem. Suppose $v \in \mathscr{F}^*_{2(M+1)}$ and let K be the associated bound constant of v in the space $\mathscr{F}^*_{2(M+1)}$. The function $F(x, y; z, q)$ has the $q \to 0$ M-term asymptotic expansion

$$F(x, y; z, q) = \exp\left(-z \int_0^1 d\xi\, v(\hat{\xi})\right)\{1 + qT_1(x, y; z) + \ldots + q^M T_M(x, y; z)\}$$
$$+ E_{M+1}(x, y; z, q). \tag{4.4}$$

For all $z \in \bar{D}$, the error term $E_{M+1}(x, y; z, q)$, has the bound

$$|E_{M+1}(x, y; z, q)| \leq \frac{q^{M+1}|z|^{M+2}\|\mu\|}{(M+1)!} \left(\frac{K^2}{4}\right)^{M+1} \exp[|z|\,\|\mu\| \exp 2(M+1)], \tag{4.5}$$

This bound is $O(q^{M+1})$ and uniform for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof. We demonstrate lemma 2 and the theorem together. Consider the small q expansion of B_n. Note that $a_n \geq 0$ for all its arguments. Use the exponential expansion identity (lemma 5(2) in OF)

$$\exp(-(zqa_n)) = \sum_{m=0}^{M} \frac{1}{m!}(-zqa_n)^m + (-zqa_n)^{M+1} \int_>^1 d^{M+1}\xi' \exp(-(zqa_n)\xi'_{M+1}). \tag{4.6}$$

With this identity B_n takes the form,

$$B_n(x, y; z, q) = (-z)^n \sum_{m=0}^{M} \frac{1}{m!} \int_>^1 d^n\xi \int d^n\mu (-zqa_n)^m\, e^{ib_n} + C_n^{M+1}(x, y; z, q) \tag{4.7}$$

where

$$C_n^{M+1}(x, y; z, q) = (-z)^n \int_>^1 d^n\xi \int d^n\mu \int_>^1 d^{M+1}\xi' (-zqa_n)^{M+1}$$
$$\times \exp(-zqa_n\xi'_{M+1} + ib_n). \tag{4.8}$$

Since $\operatorname{Re} z \geq 0$, $C_n^{M+1} = O(q^{M+1})$. With decomposition (4.7) of B_n, the series (3.6) becomes

$$F(x, y; z, q) = 1 + \sum_{n=1}^{\infty} \left((-z)^n \sum_{m=0}^{M} \frac{1}{m!} \int_>^1 d^n\xi \int d^n\mu (-zqa_n)^m\, e^{ib_n} + C_n^{M+1}(x, y; z, q)\right). \tag{4.9}$$

Simple estimates show that the $n=1\sim\infty$ summation of the $M+2$ terms in the large brackets in (4.9) are individually absolutely convergent. As an example, consider the sum of the terms C_n^{M+1}. This sum constitutes the remainder E_{M+1}. Let us majorise the C_n^{M+1} series. If $1\geqslant\xi_1\geqslant\ldots\geqslant\xi_n\geqslant 0$, the function a_n has the estimate

$$a_n\leqslant\frac{n}{4}\sum_{l=1}^{n}k_l^2, \tag{4.10}$$

see OF Lemma 5(2). Thus, since $v\in\mathscr{F}^*_{2(M+1)}$ one has

$$\int d^n|\mu|(a_n)^{M+1}\leqslant\left(\frac{n}{4}\right)^{M+1}\int d^n|\mu|\left(\sum_{l=1}^{n}k_l^2\right)^{M+1}$$
$$\leqslant\|\mu\|^n\left(\frac{n^2K^2}{4}\right)^{M+1}. \tag{4.11}$$

Furthermore, for $z\in\bar{D}$

$$|\exp(-zqa_n\xi'_{M+1}+\mathrm{i}b_n)|\leqslant 1, \tag{4.12}$$

for all k_i, ξ_i, and ξ'_{M+1}. Together (4.11), (4.12) and (4.8) give

$$|C_n^{M+1}(x,y;z,q)|\leqslant\frac{q^{M+1}|z|^{M+1}(|z|\,\|\mu\|)^n}{(M+1)!n!}\left(\frac{n^2K^2}{4}\right)^{M+1}. \tag{4.13}$$

The sum over $n=1\sim\infty$ of the right-hand side of (4.13) is finite for all $|z|<\infty$. In particular, one readily finds

$$|E_{M+1}(x,y;z,q)|\leqslant\sum_{n=1}^{\infty}|C_n^{M+1}(x,y;z,q)|$$
$$\leqslant\frac{q^{M+1}|z|^{M+2}\|\mu\|}{(M+1)!}\left(\frac{K^2}{4}\right)^{M+1}\exp(|z|\,\|\mu\|\exp 2(M+1)). \tag{4.14}$$

The last inequality in (4.14) has employed the estimate

$$\sum_{n=1}^{\infty}\frac{s^n n^j}{n!}\leqslant s\,\mathrm{e}^{s\exp j},\qquad 0\leqslant s<\infty, \tag{4.15}$$

where $j=1,2,\ldots$. Inequality (4.14) establishes bound (4.5) for the error term E_{M+1}.

Consider next the coefficient function sums. Apply estimate (3.11) to the individual terms in the sum (3.2). Obviously,

$$\left|\frac{(-z)^{n+m}}{m!}\int_{>}^{1}d^n\xi\int d^n\mu(a_n)^m\,\mathrm{e}^{\mathrm{i}b_n}\right|\leqslant\frac{|z|^{n+m}\|\mu\|^n}{m!n!}\left(\frac{n^2K^2}{4}\right)^{m+1}. \tag{4.16}$$

Summing (4.16) and again utilising inequality (4.15) leads to the bound (4.3) for $T_m(x,y;z)$. The series (4.2) defines an entire function of z because the series is absolutely convergent for all $z\in\mathbb{C}$. In fact (4.16) implies that the series (4.2) is uniformly convergent for all x, y. Each term in series (4.2) is continuous in $\mathbb{R}^d\times\mathbb{R}^d$, so it follows that the $T_m(x,y;z)$ is jointly continuous in x and y.

The semiclassical expansion (4.4) is an immediate consequence of the allowed rearrangement of the sums in (4.9).

In conclusion it is helpful to give a number of interpretative comments about the results found above. Bound estimates (4.3) and (4.5) show that $|T_m|$ and $|E_{M+1}|$ vanish if either $\|\mu\| \to 0$ (the weak interaction limit) or $|z| \to 0$ (the high-temperature/short-time limit). Equation (4.2) provides us with a definition of the coefficient functions T_m for all allowed m. Summing series (4.2) may be tedious. An efficient determination of the functions T_m proceeds as follows. Consider the formal asymptotic expansion

$$\ln F(x, y; z, q) \sim \sum_{m=1}^{\infty} q^m S_m(x, y; z). \tag{4.17}$$

The T_m are then predicted in terms of the coefficient functions S_m by the cummulant formulae

$$T_1 = S_1, \qquad T_2 = S_2 + \tfrac{1}{2}S_1^2, \qquad T_3 = S_3 + S_1 S_2 + \tfrac{1}{6}S_1^3, \tag{4.18}$$

etc. As m increases there are a large number of ways in which powers of $v(x)$ and its partial derivatives may be put together to form S_m and T_m. This is in essence a problem in combinatorics. Fujiwara *et al* (1982) have solved this problem by showing that the formulae for S_m may be constructed from a finite sum of fully connected planar graphs. Thus one can obtain S_m and T_m directly in a closed algebraic form without having to sum (4.2) or solve recursion relations. For example, the formula for $T_1(x, y; z)$ is

$$\begin{aligned} T_1(x, y; z) = &-z^2 \int_0^1 d\xi_1\, \xi_1(1-\xi_1)\Delta v(\hat{\xi}_1) \\ &+ z^3 \int_0^1 \int_0^1 d\xi_1\, d\xi_2\, \xi_<(1-\xi_>)\nabla v(\hat{\xi}_1)\cdot\nabla v(\hat{\xi}_2) \end{aligned} \tag{4.19}$$

with diagonal value

$$T_1(x, x; z) = -\tfrac{1}{6}z^2\Delta v(x) + \tfrac{1}{12}z^3(\nabla v(x))^2. \tag{4.20}$$

Upon first inspection it may appear artificial to include $\exp z \int_0^1 d\xi\, v(\hat{\xi})$ as a multiplier in front of the coefficient T_m. However, the sum over n in (3.2) always produces a compensating exponential factor such that simple closed formulae for T_m like expression (3.19) emerge. In fact, T_m is a polynomial in z whose least power is $m+1$ and whose greatest power is 3^m. For additional expressions for S_m see Fujiwara *et al* (1982).

The traditional form of the Wigner–Kirkwood expansion is obtained if we set $x = y$ and $z = \beta > 0$ in (4.4) and (1.4),

$$\begin{aligned} U(x, x; \beta, q) \sim &\frac{1}{(4\pi q\beta)^{d/2}} \exp(-\beta v(x)) \\ &\times\{1 + qT_1(x, x; \beta) + \ldots + q^M T_M(x, x; \beta)\}. \end{aligned} \tag{4.21}$$

This clearly is a semiclassical expansion in the variable q. If on the other hand we examine the time evolution kernels one must set $z = \mathrm{i}t/\hbar$. The polynomial z dependence in $T_m(x, y; z)$ means that these terms do not vanish as $\hbar \to 0$. Specifically the error term is (for $t > 0$)

$$|E_{M+1}| \leq \left(\frac{\hbar^2}{2m}\right)^{M+1} \left(\frac{t}{\hbar}\right)^{M+2} \frac{\|\mu\|}{(M+1)!} \left(\frac{K^2}{4}\right)^{M+1} \exp\left(\frac{t}{\hbar}\|\mu\| \exp 2(M+1)\right). \tag{4.22}$$

If $\hbar$ is kept at its physical value, then $|E_{M+1}| \to 0$ as the mass parameter $m \to \infty$. Thus for time evolution kernels the asymptotic expansion implied by the theorem for $F(x, y; z, q)$ is a large mass expansion.

The salient features of the approximation (4.4) for $F(x, y; z, q)$ that our analysis exposes and which are responsible for its widespread use in applications are the following.

(i) The expansion gives a rigorous non-perturbative approximation for the coordinate space kernels associated with the operators e^{-zH}, $z \in \bar{D}$.

(ii) The existence of an M-term asymptotic expansion for small q only requires that the potential have $2(M+1)$ bounded derivatives—no other feature (such as decay in $|x|$) of the potential is assumed.

(iii) The expansion (i.e., coefficient functions and error term estimate) is uniform in x and y. Furthermore, time and inverse temperature are treated on the same analytic footing. The expansion is uniform in z for z confined to any compact subset of $\bar{D}$.

(iv) One may differentiate (4.4) with respect to either z or x, y and the resultant identity is an $O(q^{M+1})$ asymptotic expansion with uniformly bounded error if $v(x)$ is sufficiently smooth. For example, if the differential operator is d/dz, then the required smoothness of the potential is $v \in \mathscr{F}^*_{2(M+2)}$. If the operator is the partial derivative, D^α_x, then one requires $v \in \mathscr{F}^*_{2(M+1)+|\alpha|}$. This stability feature of the expansion means that one may determine the short-time or high-temperature behaviour of nearly all observables (self-adjoint operators) of interest in quantum mechanics.

Some of the many practical applications of the Wigner-Kirkwood expansion can be found in the review of Singh and Sinha (1981). Because of its central role in dynamics most of the mathematical literature has concentrated in obtaining the $\hbar \to 0$ limit for the fundamental solution of the time-dependent Schrödinger equation defined by $H_{(x)}$. A particularly detailed treatment of small $\hbar$ approximations for $U(x, y; it/\hbar, \hbar^2/2m)$ has been given by Fujiwara (1979, 1980). Fujiwara's expansions assume the $\hbar \to 0$ asymptotic structure for the time evolution operator introduced by Birkhoff (1933) and Maslov and Fedoriuk (1981). Recently Schrader and Taylor (1984) have derived the $\hbar \to 0$ limit of Tr $e^{-\beta H}$ that corresponds to the dx integral of (4.21) for quantum systems whose Hamiltonian $H_{(x)}$ has support on a compact manifold.

Acknowledgment

This research has been supported in part by the US Department of Energy and the Natural Sciences and Engineering Research Council of Canada.

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